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Strict *MV*-algebras

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INTRODUCTION

The aim of this paper is to go deeper into the questions of *MV*-algebras provided with square roots. Höhle proposed a classification of *MV*-algebras with square roots, introducing the strict *MV*-algebras and proving that every *MV*-algebra with square roots is exactly a Boolean algebra, or a strict algebra, or isomorphic to the product of both. Due to the their important role we propose a characterization of strict *MV*-algebras proving that the following concepts are equivalent: (i) strict *MV*-algebras, (ii) 2-atomless *MV*-algebras. Moreover we show that a strict *MV*-algebra contains for every $n \in N$, $n \geq 0$, the cyclic elements of order 2^{n+1} and we find again the result of Höhle that their generated subalgebra is the set of rational dyadic numbers. Then, referring to the homomorphisms between *MV*-algebras, we prove that the homomorphic image of a strict *MV*-algebra is still a strict *MV*-algebra.

1. PRELIMINARIES

Let A be an *MV*-algebra; i.e., a nonempty set with two binary operations $\oplus, \odot$, one unary operation $^{-}$, and two special elements $0, 1$ such that $(A, \oplus, 0)$ and $(A, \odot, 1)$ are commutative semigroups with identity, $x \oplus \bar{x} = 1$, $x \odot \bar{x} = 0$, $\bar{0} = 1$, $\overline{(x \oplus y)} = \bar{x} \odot \bar{y}$, $\overline{x \odot y} = \bar{x} \oplus \bar{y}$, $\bar{\bar{x}} = x$.

Moreover, defining $\vee$ and $\wedge$ by the equalities $x \vee y = x \oplus \bar{x} \odot y$ and $x \wedge y = (x \oplus \bar{y}) \odot y$, with $0, 1$, A is a distributive lattice with last element 0 and greatest element 1 under the ordering $x \leq y$ if $x \wedge y = x$. We call A complete if as a lattice A is complete [1, 2, 4, 5, 7].

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Let A and B be *MV*-algebras. A function $h\colon A \to B$ is a *homomorphism* iff for each $x, y \in A$ the following conditions are satisfied: $h(0) = 0$; $h(x \oplus y) = h(x) \oplus h(y)$; $h(\bar{x}) = \overline{h(x)}$. It is easy to check that $h(x \wedge y) = h(x) \wedge h(y)$ and $h(x \vee y) = h(x) \vee h(y)$. An *ideal* of an *MV*-algebra A is a subset I of A satisfying the following conditions: $0 \in I$; if $x \in I$ and $y \leq x$ then $y \in I$; if $x \in I$ and $y \in I$ then $x \oplus y \in I$. Every ideal I of an *MV*-algebra A determines a congruence relation $\equiv_I$ on A by $x \equiv_I y$ iff $(x \odot \bar{y}) \oplus (\bar{x} \odot y) \in I$. For every $x \in A$, the equivalence class of x with respect to $\equiv_I$ is denoted by x/I and the quotient set $A/\equiv_I$ by A/I. The set A/I, equipped with the operations $(\overline{x/I}) := \bar{x}/I$, $x/I \oplus y/I := x \oplus y/I$, $x/I \odot y/I := x \odot y/I$ and the constants $0/I$ and $1/I$, became an *MV*-algebra, called the *quotient algebra of A by the ideal I*.

The correspondence $x \mapsto x/I$ defines a homomorphism h_I from A onto the quotient algebra A/I, which is called the *natural homomorphism* from A onto A/I. It is remarkable that for any arbitrary *MV*-algebra A and for all $x, y, z \in A$ we have $z \odot x \leq y$ iff $z \leq \bar{x} \oplus y$.

Chang [5] proved that there is a one-to-one correspondence between linearly ordered *MV*-algebras and linearly ordered abelian groups.

Mundici [7] extended such a correspondence to a functor Γ from lattice-ordered abelian groups with a strong unit to *MV*-algebras. Indeed let G be an abelian lattice-ordered group with strong unit u. Then by definition,

$$\Gamma(G, u) = [0, u] = \{x \in G \mid 0 \leq x \leq u\}$$

is the *MV*-algebra A obtained by equipping the unit interval $[0, u]$ of G with the following operations: $x \oplus y = (x + y) \wedge u$, $x \odot y = (x + y - u) \vee 0$, $\bar{x} = u - x$, and identifying u with the unit element 1 of A.

We recall now the basic definitions and the classification of *MV*-algebras with square roots by Höhle [6].

DEFINITION 1.1. An *MV*-algebra A has *square roots* if there exists a unary operation $S\colon A \to A$ provided with the following properties:

P1. $S(x) \odot S(x) = x \ \ \forall x \in A$

P2. $\forall x \in A, \forall y \in A, \ y \odot y \leq x \Rightarrow y \leq S(x)$.

DEFINITION 1.2. An *MV*-algebra with square roots is called *strict* iff $S(0) = \overline{S(0)}$.

A Boolean algebra with at least two elements is never a strict *MV*-algebra, while the real unit interval $[0, 1]$ provided with Lukasiewicz's arithmetic conjunction is an example of a strict *MV*-algebra.

Now the classification of Höhle for the MV-algebras with square roots is represented by the following:

THEOREM 1.1. *For every MV-algebra A with square roots one of the following assertions holds*

(1) *A is a Boolean algebra*;

(2) *A is a strict MV-algebra*;

(3) *A is isomorphic to a product of a Boolean algebra and a strict MV-algebra*.

DEFINITION 1.3 [3]. An MV-algebra A is *strongly atomless* if for every $x > 0$ there exists $z \in A$ such that

(1) $0 < z < x$,

(2) $x \odot \bar{z} \leq z$.

Höhle proves, according to the terminology of Belluce [2],

THEOREM 1.2. *Let A be an MV-algebra with square roots. If A is strongly atomless, then A is strict.*

THEOREM 1.3. *Let A be a complete MV-algebra. If A is infinite and locally finite, then A is strict.*

THEOREM 1.4. *Let A be a complete MV-algebra. Then the following assertions are equivalent*:

(i) *A is strict*;

(ii) *A is divisible in the sense of Belluce*;

(iii) *A is strongly atomless*;

(iv) *A is injective in the category of MV-algebras*.

In the next section we strengthen this theorem by referring to every MV-algebra without the hypothesis of completeness.

Now we give the following definitions and theorem by Torrens [8].

DEFINITION 1.4. *A finite chain* is the set $C_n\{\frac{i}{n},\ i \in \{0, 1, \ldots, n\}\}$ for every $n \in N$.

Of course C_n is a subalgebra of the MV-algebra $[0, 1]$.

DEFINITION 1.5. Let A be an MV-algebra and let $0 < n \in N$. An element a in A is called cyclic of order $n > 0$, if it satisfies the following assertion: $(n-1)\bar{a} = a$.

In any MV-algebra the unique cyclic element of order 1 is 0.

THEOREM 1.5 [8]. *An MV-algebra A has a cyclic element of order n if and only if A contains a copy of* C_n.

2. CHARACTERIZATION OF THE STRICT *MV*-ALGEBRAS

In order to characterize the strict *MV*-algebras we premise the following definition:

DEFINITION 2.1. An *MV*-algebra is called 2-*atomless* if for every $x > 0$ there exists $z \in A$ such that

(1) $0 < z < x$

(2) $x \odot \bar{z} = z$.

LEMMA 2.1. *Let A be an MV-algebra. A is strict if and only if*

(1) $\forall x \in A, \exists \alpha \in A : x = \alpha \oplus \alpha$,

(2) $\exists$ *an element* $\xi \in A$ *such that* $\xi = \bar{\xi}$.

Proof. Let A be strict. Then A has square roots; i.e., there exists a unary operation S: $A \to A$ such that $S(x) \odot S(x) = x$ for every $x \in A$ and $S(0) = \overline{S(0)}$. Thus the condition (2) is proved.

In order to verify condition (1), we infer from axiom **P1**:

$$\overline{S(\bar{x})} \oplus \overline{S(\bar{x})} = \overline{S(\bar{x}) \odot S(\bar{x})} = \bar{\bar{x}} = x.$$

Vice versa, let us assume the validity of (1) and (2). We show that $\xi \oplus \alpha$ is the square root of x, whenever $\alpha \oplus \alpha = x$. We choose $\alpha \in A$ with $x = \alpha \oplus \alpha$; then we infer from condition (2),

$$\begin{aligned} \xi + \alpha - x &= u - x + (\alpha - \xi) = \left[(u - (\alpha + \alpha)) \vee 0\right] + (\alpha - \xi) \\ &= \left[((\xi - \alpha) + (\xi - \alpha)) \vee 0\right] + (\alpha - \xi) \\ &= (\xi - \alpha) \vee (\alpha - \xi) = |\xi - \alpha| \geq 0; \end{aligned}$$

hence the important relation

$$x \leq \xi + \alpha$$

holds. Further we observe

$$\begin{aligned} (\xi \oplus \alpha) \odot (\xi \oplus \alpha) &= \left(\left[((\xi + \alpha) \wedge u) + ((\xi + \alpha) \wedge u)\right] - u\right) \vee 0 \\ &= \left[(\xi + \xi + \alpha + \alpha) \wedge (\xi + \alpha + u) \wedge (u + u)\right] - u \\ &= (\alpha + \alpha) \wedge (\alpha + \xi) \wedge u = x \wedge (\alpha + \xi) = x; \end{aligned}$$

$$\begin{aligned} y \odot y \leq x &\Leftrightarrow y + y - u \leq \alpha + \alpha \\ &\Leftrightarrow 0 \leq ((\xi + \alpha) - y) + ((\xi + \alpha) - y) \\ &\Leftrightarrow 0 \leq (\xi + \alpha) - y \Leftrightarrow y \leq \xi \oplus \alpha. \end{aligned}$$

Hence the axioms **P1** and **P2** are satisfied. ∎

THEOREM 2.1. *Let A be an MV-algebra. Then the following assertions are equivalent*:

(1) *A is 2-atomless.*

(2) *A such that $\forall x \in A$, $\exists \alpha \in A : x = \alpha \oplus \alpha$, and $\exists$ an element $\xi = \bar{\xi}$.*

Proof. If A is 2-atomless; i.e., for every $x > 0$ there exists $z \in A$ such that $0 < z < x$ and $x \odot \bar{z} = z$, we have $x \odot \bar{z} \oplus z = z \oplus z$, then $x \vee z = z \oplus z$, i.e., $x = z \oplus z$. Moreover for $x = 1$ we have $\bar{z} = z$ and z is a self-negate element of A.

Vice versa, if the assertion (2) holds, then $\alpha \oplus \alpha = x$ implies $x \leq \alpha + \xi$. Hence $z := \alpha \wedge \xi$ is also a solution of $x = \alpha \oplus \alpha$. Therefore, for $z = \alpha \wedge \xi$ we get $\bar{z} \odot x = z$. Hence A is 2-atomless. ∎

Combining the foregoing theorems we obtain a characterization of the strict MV-algebras.

THEOREM 2.2. *Let A be an MV-algebra. Then the following assertions are equivalent*:

(1) *A is strict*

(2) *A is 2-atomless*

EXAMPLE 1. A strict, simple, and not complete MV-algebra is given by the MV-algebra D of all rational dyadic numbers subalgebra of the MV-algebra $[0,1]$:

$$D = \bigcup_{n \in N} \left\{ \frac{i}{2^n},\ i \in \{0, 1, \ldots, 2^n\} \right\}.$$

EXAMPLE 2. A strict, not complete, and not simple MV-algebra is given by the MV-algebra $A = \Gamma(R \times R, (1,0))$ where $R \times R$ is the group of lexicographic product of R by R.

Let $S_k(x)$ denote $\underbrace{S(S(\ldots S(x)))}_{k \text{ times}}$, then we have

LEMMA 2.2. *Let A be a strict MV-algebra. Then for every $x \in A$ and for every $h, k \in N$, with $h \leq k$, the following holds*

$$2^h \overline{S_k(x)} = \overline{S_{k-h}(x)}.$$

Proof. If $h = 1$ we have $2\overline{S_k(x)} = \overline{S_k(x)} \oplus \overline{S_k(x)} = \overline{S_k(x) \odot S_k(x)} = \overline{S_{k-1}(x)}$, because S is a root.

Then, iterating this procedure, we have $2^h\overline{S_k(x)} = 2^{h-1}\overline{S_k(x)} \oplus 2^{h-1}\overline{S_k(x)} = \overline{S_{k-h+1}(x)} \oplus \overline{S_{k-h+1}(x)} = \overline{S_{k-h+1}(x) \odot S_{k-h+1}} = \overline{S_{k-h}(x)}$. ∎

THEOREM 2.3. *Let A be a strict MV-algebra. Then, for every $n \in N$, $n \geq 0$, A contains the cyclic element of order* 2^{n+1}.

Proof. Since A is strict, then there exists $\xi \in A$ such that $\xi = \bar{\xi}$ and ξ is the cyclic element of order 2.

We claim that $S_n(\xi)$ is the cyclic element of order 2^{n+1}, i.e.,

$$(2^{n+1} - 1)\overline{S_n(\xi)} = S_n(\xi).$$

Indeed, $(2^{n+1} - 1)\overline{S_n(\xi)} = 1\overline{S_n(\xi)} \oplus 2\overline{S_n(\xi)} \oplus \cdots \oplus 2^n\overline{S_n(\xi)}$. Then by Lemma 2.2, we have

$$\begin{aligned}
(2^{n+1} - 1)\overline{S_n(\xi)} &= \overline{S_n(\xi)} \oplus \overline{S_{n-1}(\xi)} \oplus \cdots \oplus \overline{S(\xi)} \oplus \xi \\
&= \overline{S_n(\xi)} \oplus \overline{S_{n-1}(\xi)} \oplus \cdots \oplus \overline{S(\xi)} \oplus S(\xi) \odot S(\xi) \\
&= \overline{S_n(\xi)} \oplus \overline{S_{n-1}(\xi)} \oplus \cdots \oplus \left(\overline{S(\xi)} \vee S(\xi)\right) \\
&= \overline{S_n(\xi)} \oplus \overline{S_{n-1}(\xi)} \oplus \cdots \oplus S(\xi).
\end{aligned}$$

In general $\overline{S_k(\xi)} \oplus S_{k-1}(\xi) = S_k(\xi)$. Hence $(2^{n+1} - 1)\overline{S_n(\xi)} = S_n(\xi)$. ∎

By the above theorem and by the existence of such cyclic elements within a strict *MV*-algebra, we get an alternative proof of a result obtained by Höhle in Theorem 6.9 [6]:

THEOREM 2.4. *Let A be a strict MV-algebra. Then A contains a copy of the MV-algebra D of all rational dyadic numbers* (*see Example* 2).

Proof. By Theorem 2.3 A contains, for every $n \in N$, $n \geq 0$, the cyclic element of order 2^{n+1}. Therefore, by Theorem 1.5, A contains a copy of $C_{2^{n+1}}$. Then A contains a copy of D. ∎

3. HOMOMORPHIC IMAGES OF STRICT *MV*-ALGEBRAS

Referring to the operation $\wedge$ in an *MV*-algebra with square roots, we can note the following result:

PROPOSITION 3.1. *Let A be an MV-algebra with square root S. Then, for every $x, y \in A$, $x \wedge \bar{x} \leq S(y)$.*

Proof. Since for every $x \in A$, $(x \wedge \bar{x})^2 = (x \wedge \bar{x}) \odot (x \wedge \bar{x}) = 0$, then by the property **P2** we get $x \wedge \bar{x} \leq S(y)$ for every $y \in A$. ∎

THEOREM 3.1. *The homomorphic image of an MV-algebra with square roots is an MV-algebra with square roots too.*

Proof. Let A be an MV-algebra with square roots, B be a further MV-algebra, and let $h: A \mapsto B$ be a surjective MV-algebra homomorphism. We show that B also has square roots and h preserves the formation of square roots:

Let $a \in A$ be given; we show that $h(S(a))$ is the square root of $h(a)$ where S denotes the formation of square roots in A. Obviously, $h(S(a)) \odot h(S(a)) = h(a)$. Further we infer from $h(c) \odot h(c) \le h(a)$ that $h(\overline{c \odot c} \oplus a) = u$. Now we apply Propositions 2.11 and 2.17 in [6],

$$\overline{c \odot c} \oplus a \le S(\overline{c \odot c} \oplus a) = \overline{c \vee S(0)} \oplus S(a) = \bar{c} \oplus S(a);$$

hence we obtain $u = h(\bar{c} \oplus S(a))$—i.e., $h(c) \le h(S(a))$. Therefore $h(S(a))$ satisfies the axioms **P1** and **P2** w.r.t. $h(a)$. ∎

THEOREM 3.2. *The homomorphic image of a strict MV-algebra is a strict MV-algebra too.*

Proof. Let A be a strict MV-algebra, h a homomorphism of A, and S the square root of A.

Then, by Theorem 3.1, there exists a square root S^* of $h(A)$ such that $S^*(h(a)) = h(S(a))$. Then $S^*(h(0)) = h(S(0))$. But $h(S(0)) = h(\overline{S(0)}) = \overline{h(S(0))} = \overline{S^*(h(0))}$, and then the MV-algebra $h(A)$ is strict. ∎

Referring to Section 1 we can also express Theorem 3.2 as follows: Strictness of MV-algebras is inherited by quotients.

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